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Coherent states for the q-deformed quantum mechanics on a circle

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Abstract
The q-deformed coherent states for a quantum particle on a circle are introduced and their properties investigated.

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1. Introduction

In spite of the fact that the first paper devoted to the quantum mechanics on a circle is most probably the article by Condon on the quantum pendulum, dated 1928 [1], the coherent states for a quantum particle on a circle have been introduced only recently [2, 3]. It is no wonder that there is also no example of quantum deformations of these states so far. We recall that the q-deformation of the standard coherent states was constructed 15 years ago in [4]. The need for such deformations is motivated, among others, by the importance of the non-deformed coherent states for the quantum mechanics on a circle, which have been already applied for example in quantum gravity [5]. In this work, we introduce the q-generalization of the coherent states for a quantum particle on a circle. We first recall the basic facts about the quantum mechanics on a circle. Consider a free particle on a circle S^1. According to [2], the best candidate to represent the position of a particle on the unit circle is the unitary operator $U = e^{i\hat{\varphi}}$ ($\hat{\varphi}$ Hermitian) satisfying the following commutation rule with the Hermitian angular momentum operator J:

$$[J, U] = U. \tag{1.1}$$

In the Hilbert space $L^2(S^1)$ of 2π-periodic functions, specified by the scalar product

$$\langle f|g\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, f^*(\varphi)g(\varphi), \tag{1.2}$$

the operators U and J act as follows:

$$Uf(\varphi) = e^{i\varphi} f(\varphi), \tag{1.3}$$

and

$$Jf(\varphi) = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\varphi}f(\varphi). \tag{1.4}$$

Consider now the eigenvalue equation

$$J|j\rangle = j|j\rangle. \tag{1.5}$$

Demanding the time-reversal invariance of representations of (1.1) we have only two possibilities left: j integer and j half-integer [2]. In this work, we restrict ourselves for simplicity to the case of integer j. Using (1.1) one finds that the operators U and $U^\dagger$ are the ladder operators such that

$$U|j\rangle = |j+1\rangle, \tag{1.6a}$$

$$U^\dagger|j\rangle = |j-1\rangle. \tag{1.6b}$$

Projecting (1.6*b*) on the eigenvector $|\varphi\rangle$ of U satisfying

$$U|\varphi\rangle = \mathrm{e}^{\mathrm{i}\varphi}|\varphi\rangle \tag{1.7}$$

we get

$$e_j(\varphi) := \langle\varphi|j\rangle = \mathrm{e}^{\mathrm{i}j\varphi}. \tag{1.8}$$

Of course, the vectors $e_j(\varphi)$ are the basis vectors in the representation space $L^2(S^1)$. We finally write down the orthogonality and completeness conditions satisfied by the vectors $|j\rangle$ of the form

$$\langle j|j'\rangle = \delta_{jj'}, \tag{1.9}$$

$$\sum_{j=-\infty}^{\infty}|j\rangle\langle j| = I. \tag{1.10}$$

2. Coherent states for the quantum mechanics on a circle

We now recall the construction of (non-deformed) coherent states $|\xi\rangle$ for a quantum particle on a circle described in [2], where these are introduced as the solution of the eigenvalue equation

$$Z|\xi\rangle = \xi|\xi\rangle, \tag{2.1}$$

where

$$Z = \exp\left(-J + \tfrac{1}{2}\right)U. \tag{2.2}$$

The convenient parametrization of ξ consistent with the form (2.2) of the operator Z is

$$\xi = \exp(-l + \mathrm{i}\alpha). \tag{2.3}$$

We point out that the parametrization (2.3) relies on the deformation of the cylinder (the phase space) specified by

$$x = \mathrm{e}^{-l}\cos\alpha, \qquad y = \mathrm{e}^{-l}\sin\alpha, \qquad z = l, \tag{2.4}$$

and then projecting the points of the obtained surface onto the (x, y) (complex) plane. Evidently, we then identify the points of the cylinder with the plane with extracted point $(0, 0)$ (the origin).

The projection of the vectors $|\xi\rangle$ onto the basis vectors $|j\rangle$ is given by

$$\langle j|\xi\rangle = \xi^{-j}\,\mathrm{e}^{-\frac{j^2}{2}}. \tag{2.5}$$

Making use of the parameters l and α we can write (2.5) in the following equivalent form:

$$\langle j|l,\alpha\rangle = \exp(lj - \mathrm{i}j\alpha)\exp\left(-\frac{j^2}{2}\right), \tag{2.6}$$

where $|l,\alpha\rangle \equiv |\xi\rangle$, with $\xi = \exp(-l + \mathrm{i}\alpha)$.

The coherent states are not orthogonal. The overlap integral is [2]

$$\langle\xi|\eta\rangle = \sum_{j=-\infty}^{\infty}(\xi^*\eta)^{-j}\,\mathrm{e}^{-j^2} = \theta_3\left(\frac{\mathrm{i}}{2\pi}\ln\xi^*\eta\,\middle|\,\frac{\mathrm{i}}{\pi}\right), \tag{2.7}$$

where θ_3 is the Jacobi theta function [6].

3. q-deformed coherent states for a particle on a circle

We first introduce a q-deformation of the algebra (1.1). The general deformation of the quantum mechanics on a circle was considered in [7]. In this work, we consider a q-deformation of the algebra (1.1) generated only by U, J_q and the identity operator, of the form

$$qUJ_q = J_qU - U, \tag{3.1}$$

where U is a unitary operator representing the position of a quantum particle on a circle. We also restrict to the case of $q > 0$. The fact that q is real follows directly from (3.1) and the assumed hermicity of J_q and unitarity of U. Now, it can be easily verified that the q-deformed angular momentum satisfying (3.1) is an element of the enveloping algebra of (1.1) such that

$$J_q = \frac{q^J - 1}{q - 1}, \tag{3.2}$$

while U remains undeformed. Clearly, J_q acts on the basis vectors $|j\rangle$ as follows:

$$J_q|j\rangle = [j]_q|j\rangle, \tag{3.3}$$

where $[j]_q = \frac{q^j-1}{q-1}$ is a quantum integer. Furthermore, using (1.4) and (3.2) we find that the action of the operator J_q in $L^2(S^1)$ is of the following form:

$$J_q f(\mathrm{e}^{\mathrm{i}\varphi}) = \mathrm{e}^{\mathrm{i}\varphi}D_q f(\mathrm{e}^{\mathrm{i}\varphi}), \tag{3.4}$$

where D_q designates the Jackson derivative defined by

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}, \tag{3.5}$$

and we utilized the fact that in the discussed case of $L^2(S^1)$ spanned by the functions $\mathrm{e}^{\mathrm{i}j\varphi}$, where j is an integer, a 2π-periodic function of φ is assumed to have the Fourier series expansion and it can be considered as a function of $\mathrm{e}^{\mathrm{i}\varphi}$. Furthermore, taking into account (3.2) we get

$$[J_q, \hat{\varphi}] = -\mathrm{i}\ln q\, J_q - \mathrm{i}\frac{\ln q}{q-1}, \tag{3.6}$$

where $\hat{\varphi} = -\mathrm{i}\ln U$. We remark that the commutator (3.6) is nontrivially defined except of the case $q = 1$ [8].

Now we are in a position to define q-deformed coherent states for the quantum mechanics on a circle. Proceeding analogously as in the case of non-deformed coherent states discussed in the previous section we define the q-deformed coherent states as a solution of the eigenvalue equation

$$Z_q|\xi\rangle_q = \xi_q|\xi\rangle_q, \tag{3.7}$$

where

$$Z_q := \exp(\mathrm{i}(\hat{\varphi} + \mathrm{i}J_q)). \tag{3.8}$$

Using (3.6) we find after some calculation

$$Z_q = \exp\left(\frac{1}{1-q}\left(\frac{1-q^{-1}}{\ln q} - 1\right)\right)\exp\left(\frac{q^{-1}-1}{\ln q}J_q\right)U. \tag{3.9}$$

Taking into account (3.7), (3.9), (3.3) and (1.6*b*) we get

$$\langle j|\xi\rangle_q = \xi_q^{-j}\exp\left(-\frac{1}{\ln q}\frac{q^j-1}{q-1}\right)\exp\left(-\frac{j}{1-q}\right). \tag{3.10}$$

Now, the form of the operator (3.8) and (2.3) indicates the following parametrization of a complex number ξ_q:

$$\xi_q = \exp(-[l]_q + \mathrm{i}\alpha), \tag{3.11}$$

where $[l]_q = \frac{q^l-1}{q-1}$, where l is a real number, is quantum l. It should be noted that for $0 < q < 1$, $[l]_q$ is an increasing function of l and it has an upper bound $1/(1-q)$ approached in the limit $l \to +\infty$. Consequently (see (2.4)), in the case of $0 < q < 1$, we identify the points of the q-deformed classical phase space with the (x, y) plane with extracted disk $x^2 + y^2 \leqslant \exp\left(-\frac{2}{1-q}\right)$. Evidently, this disk reduces to the point $(0, 0)$, i.e. the origin, in the limit $q \to 1$ referring to the non-deformed case. In the case of $q > 1$, $[l]_q$ is also an increasing function and it has a lower bound $-1/(q-1)$ reached in the limit $l \to -\infty$. Therefore, in this case we identify the points of the deformed classical phase space with the disk $x^2 + y^2 < \exp\left(\frac{2}{q-1}\right)$, with extracted point $(0, 0)$. Obviously, in the limit $q \to 1$ corresponding to the non-deformed case, the disk is simply the plane (x, y) with extracted origin.

Using (3.11) we can write (3.10) in the form

$$\langle j|l,\alpha\rangle_q = \exp(-\mathrm{i}j\alpha)\exp\left(-\frac{1}{\ln q}\frac{q^j-1}{q-1}\right)\exp\left(-\frac{q^l}{1-q}j\right), \tag{3.12}$$

where $|l,\alpha\rangle_q \equiv |\xi\rangle_q$, and ξ_q referring via (3.7) to $|\xi\rangle_q$, is given by (3.11). Clearly, relation (2.6) refers to the limit $q \to 1$ in (3.12). We recall that the functions (2.6) span the Bargmann representation [2]. The problem of finding the Bargmann representation in the discussed case of the q-deformed coherent states for a quantum particle on a circle is complicated and it will be discussed in a separate work.

The coherent states are not orthogonal. Indeed, making use of (3.12) and (1.10) we find

$${}_q\langle l,\alpha|h,\beta\rangle_q = \sum_{j=-\infty}^{\infty}\exp(\mathrm{i}(\alpha-\beta)j)\exp\left(-\frac{2}{\ln q}\frac{q^j-1}{q-1}\right)\exp\left(-\frac{q^l+q^h}{1-q}j\right). \tag{3.13}$$

In the limit $q \to 1$ we recover, from (3.13), the following formula [2]:

$$\langle l,\alpha|h,\beta\rangle = \theta_3\left(\frac{1}{2\pi}(\alpha-\beta) - \frac{l+h}{2}\frac{\mathrm{i}}{\pi}\,\middle|\,\frac{\mathrm{i}}{\pi}\right). \tag{3.14}$$

It follows immediately from (3.13) that the discussed coherent states are not normalized. Namely, we have

$${}_q\langle l,\alpha|l,\alpha\rangle_q = \sum_{j=-\infty}^{\infty}\exp\left(-\frac{2}{\ln q}\frac{q^j-1}{q-1}\right)\exp\left(-\frac{2q^l}{1-q}j\right). \tag{3.15}$$

For $q = 1$ this relation reduces to

$$\langle l, \alpha | l, \alpha\rangle = \sum_{j=-\infty}^{\infty} \mathrm{e}^{2lj}\,\mathrm{e}^{-j^2} = \theta_3\left(\frac{\mathrm{i}l}{\pi}\middle|\frac{\mathrm{i}}{\pi}\right). \tag{3.16}$$

One can easily check that the series (3.15) is convergent for arbitrary (positive) q and (finite) l. Therefore, by virtue of the Schwartz's inequality the series (3.13) is also convergent for arbitrary q and l. We finally write down the formula on the coherent states in $L^2(S^1)$ following directly from (1.8), (1.10) and (3.12) such that

$$\langle\varphi|l,\alpha\rangle_q = \sum_{j=-\infty}^{\infty} \exp(\mathrm{i}j(\varphi-\alpha)) \exp\left(-\frac{1}{\ln q}\frac{q^j-1}{q-1}\right)\exp\left(-\frac{q^l}{1-q}j\right). \tag{3.17}$$

In the limit $q \to 1$ this formula takes the form

$$\langle\varphi|l,\alpha\rangle = \theta_3\left(\frac{1}{2\pi}(\varphi-\alpha-\mathrm{i}l)\middle|\frac{\mathrm{i}}{2\pi}\right). \tag{3.18}$$

Note that in view of (3.14), (3.16) and (3.18) the series from the right-hand side of (3.13), (3.15) and (3.17), respectively, can be regarded as q-deformations of the Jacobi theta functions.

4. q-deformed coherent states and the classical phase space

In this section, we discuss the parametrization (3.11) of the deformed phase space in more detail. Consider the expectation value of the deformed angular momentum operator J_q given by (3.2). On using (1.10), (3.3) and (3.12) we arrive at the following relation:

$$\frac{{}_q\langle l,\alpha|J_q|l,\alpha\rangle_q}{{}_q\langle l,\alpha|l,\alpha\rangle_q} = \frac{\sum_{j=-\infty}^{\infty}\frac{q^j-1}{q-1}\exp\left(-\frac{2}{\ln q}\frac{q^j-1}{q-1}\right)\exp\left(-\frac{2q^l}{1-q}j\right)}{\sum_{j=-\infty}^{\infty}\exp\left(-\frac{2}{\ln q}\frac{q^j-1}{q-1}\right)\exp\left(-\frac{2q^l}{1-q}j\right)}. \tag{4.1}$$

It follows from numerical calculations that there are large regions of the phase space parametrized by q and l such that

$$\frac{{}_q\langle l,\alpha|J_q|l,\alpha\rangle_q}{{}_q\langle l,\alpha|l,\alpha\rangle_q} \approx [l]_q, \tag{4.2}$$

where the approximation is very good. For example, for $q = 0.5$ and $l \geqslant 0.3$ the maximal relative error is of order 1%. The fact that (4.2) is not valid for arbitrary q and l is not surprising. We only recall that in the case of the coherent states for the quantum mechanics on a sphere [9] we have a condition $|\boldsymbol{l}| \geqslant 10$, where $|\boldsymbol{l}|$ is the norm of the vector $\boldsymbol{l}$ of the classical angular momentum parametrizing the coherent states, ensuring the (approximate) coincidence of the average of the angular momentum operator and $\boldsymbol{l}$. In our opinion, the meaning of the approximate relations like (4.2) is that the coherent states are as close as possible to the classical phase space. We conclude that the parameter $[l]_q$ in (3.11) can be regarded as a deformed version of the classical angular momentum. Let us finally recall that in the limit $q \to 1$, when (4.1) reduces to

$$\begin{aligned}\frac{\langle l,\varphi|J|l,\varphi\rangle}{\langle l,\varphi|l,\varphi\rangle} &= \frac{1}{2\theta_3\left(\frac{\mathrm{i}l}{\pi}\middle|\frac{\mathrm{i}}{\pi}\right)}\frac{\mathrm{d}}{\mathrm{d}l}\theta_3\left(\frac{\mathrm{i}l}{\pi}\middle|\frac{\mathrm{i}}{\pi}\right)\\ &= l - 2\pi\sin(2l\pi)\\ &\quad\times\sum_{n=1}^{\infty}\frac{\exp(-\pi^2(2n-1))}{(1+\exp(-\pi^2(2n-1))\exp(2\mathrm{i}l\pi))(1+\exp(-\pi^2(2n-1))\exp(-2\mathrm{i}l\pi))},\end{aligned} \tag{4.3}$$

we have a perfect approximation of the classical phase space for arbitrary l [2]. Namely, the maximal error of (4.2) is of order 0.1%, and we have the *exact* equality in the case with l integer or half-integer.

We now examine the role of the parameter α in the parametrization (3.11). Taking into account (1.10), (1.6*a*) and (3.12) we find

$$\frac{{}_q\langle l,\alpha|U|l,\alpha\rangle_q}{{}_q\langle l,\alpha|l,\alpha\rangle_q} = \mathrm{e}^{\mathrm{i}\alpha}\frac{\exp\left(\frac{q^l-1}{q-1}-\frac{1}{\ln q}-\frac{1}{1-q}\right)\sum_{j=-\infty}^{\infty}\exp\left(-\frac{1+q}{\ln q}\frac{q^j-1}{q-1}\right)\exp\left(-\frac{2q^l}{1-q}j\right)}{\sum_{j=-\infty}^{\infty}\exp\left(-\frac{2}{\ln q}\frac{q^j-1}{q-1}\right)\exp\left(-\frac{2q^l}{1-q}j\right)}. \tag{4.4}$$

In the limit $q \to 1$ this formula takes the form

$$\frac{\langle l,\alpha|U|l,\alpha\rangle}{\langle l,\alpha|l,\alpha\rangle} = \mathrm{e}^{-\frac{1}{4}}\,\mathrm{e}^{\mathrm{i}\alpha}\frac{\theta_2\left(\frac{\mathrm{i}l}{\pi}\middle|\frac{\mathrm{i}}{\pi}\right)}{\theta_3\left(\frac{\mathrm{i}l}{\pi}\middle|\frac{\mathrm{i}}{\pi}\right)} = \mathrm{e}^{-\frac{1}{4}}\,\mathrm{e}^{\mathrm{i}\alpha}\frac{\theta_3\left(l+\frac{1}{2}\middle|\mathrm{i}\pi\right)}{\theta_3(l|\mathrm{i}\pi)}. \tag{4.5}$$

Proceeding analogously as in [2] we define the relative expectation value

$${}_q\langle\langle U\rangle\rangle_{(l,\alpha)} = \frac{{}_q\langle U\rangle_{(l,\alpha)}}{{}_q\langle U\rangle_{(l,0)}}, \tag{4.6}$$

where ${}_q\langle U\rangle_{(l,\alpha)} = {}_q\langle l,\alpha|U|l,\alpha\rangle_q/{}_q\langle l,\alpha|l,\alpha\rangle_q$. Hence,

$${}_q\langle\langle U\rangle\rangle_{(l,\alpha)} = \mathrm{e}^{\mathrm{i}\alpha}. \tag{4.7}$$

Therefore, the parameter α labelling the coherent states can be identified with the classical angle.

We now study the distribution of eigenvectors $|j\rangle$s of the operator J_q in the normalized coherent state. We recall that in the non-deformed case this is the distribution of energies of a quantum particle moving freely in a (unit) circle [2]. The computer simulations indicate that the function (see (3.12) and (3.15))

$$p_{l,q}(j) = \frac{|\langle j|l,\alpha\rangle_q|^2}{{}_q\langle l,\alpha|l,\alpha\rangle_q} = \frac{\exp\left(-\frac{2}{\ln q}\frac{q^j-1}{q-1}\right)\exp\left(-\frac{2q^l}{1-q}j\right)}{\sum_{j=-\infty}^{\infty}\exp\left(-\frac{2}{\ln q}\frac{q^j-1}{q-1}\right)\exp\left(-\frac{2q^l}{1-q}j\right)}, \tag{4.8}$$

which gives the probability of finding the system in the state $|j\rangle$ when the system is in the normalized coherent state $|l,\alpha\rangle_q/\sqrt{{}_q\langle l,\alpha|l,\alpha\rangle_q}$, has the same behaviour as in the non-deformed case, that is $p_{l,q}(x)$ is peaked at $x = l$. However, in opposition to the distribution referring to the non-deformed coherent states, when $q = 1$, which is the 'discrete' Gaussian distribution of the form [2]

$$p_l(j) = \frac{|\langle j|l,\alpha\rangle|^2}{\langle l,\alpha|l,\alpha\rangle} = \frac{\mathrm{e}^{2lj}\,\mathrm{e}^{-j^2}}{\theta_3\left(\frac{\mathrm{i}l}{\pi}\middle|\frac{\mathrm{i}}{\pi}\right)} \approx \frac{\exp(-(j-l)^2)}{\sqrt{\pi}}, \tag{4.9}$$

where the approximation is very good, the distribution (4.8) is asymmetrical one (see figure 1).

We finally discuss the probability density $p_{(l,\alpha),q}(\varphi)$ for the coordinates in the normalized coherent state $|l,\alpha\rangle_q/\sqrt{{}_q\langle l,\alpha|l,\alpha\rangle_q}$ of the form (see (3.17))

$$p_{(l,\alpha),q}(\varphi) = \frac{|\langle\varphi|l,\alpha\rangle_q|^2}{{}_q\langle l,\alpha|l,\alpha\rangle_q} = \frac{\left|\sum_{j=-\infty}^{\infty}\exp(\mathrm{i}j(\varphi-\alpha))\exp\left(-\frac{1}{\ln q}\frac{q^j-1}{q-1}\right)\exp\left(-\frac{q^l}{1-q}j\right)\right|^2}{\sum_{j=-\infty}^{\infty}\exp\left(-\frac{2}{\ln q}\frac{q^j-1}{q-1}\right)\exp\left(-\frac{2q^l}{1-q}j\right)}. \tag{4.10}$$

As with the non-deformed case (see (3.18) and (3.16)) the function $p_{(l,\alpha),q}(\varphi)$ is peaked at $\varphi = \alpha$ (see figure 2). Therefore, the parameter α in (3.11) can be really regarded as the classical angle.

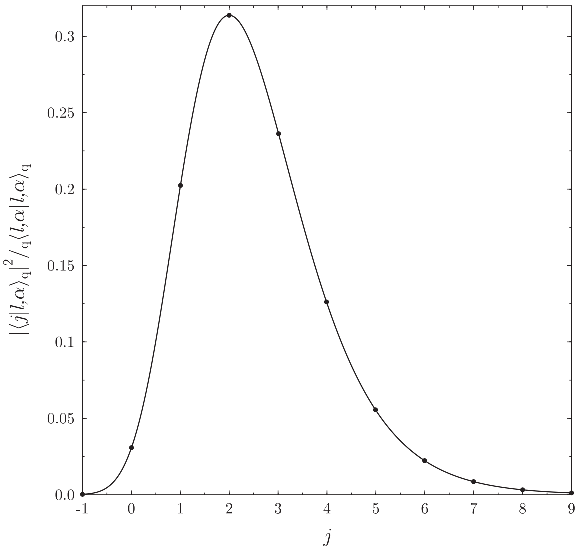

Figure 1. The plot of $p_{l,q}(j)$ (see (4.8)), where $q = 0.5$ and $l = 2$. The maximum is reached at $j_{\max} = l$.

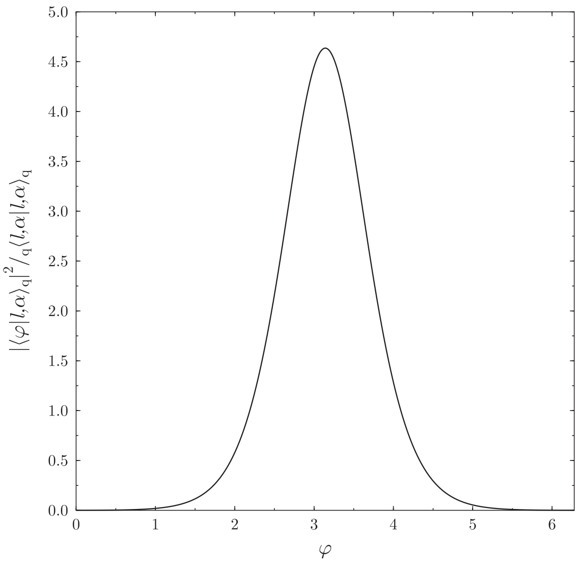

Figure 2. The probability density $p_{(l,\alpha),q}(\varphi)$ given by (4.10), where $q = 0.5$, $l = 1$ and $\alpha = \pi$. The probability density is peaked at $\varphi = \alpha$.

5. A generalization of the q-deformed coherent states

We finally study a generalization of the q-deformed coherent states discussed above arising from taking into consideration the so called 'squeezed states' introduced in [10]. These states amount a generalization of the non-deformed coherent states for the quantum mechanics on a circle introduced in [2] and can be regarded as a version of the coherent states on a circle introduced in [3]. Namely, they can be defined as a solution of the eigenvalue equation generalizing (2.1) such that [10]

$$Z(s)|\xi\rangle_s = \xi|\xi\rangle_s, \tag{5.1}$$

where

$$Z(s) = \exp\left(-s\left(J - \tfrac{1}{2}\right)\right)U, \tag{5.2}$$

and $s > 0$ is a real parameter. Clearly, the case of $s = 1$ corresponds to the coherent states discussed in section 2. An attempt to provide a physical interpretation of this dimensionless parameter was made in [3] and [11], where it is suggested that it controls the ratio of spatial width of the coherent states to the length of the circle. The counterparts of relations (2.5) and (2.7) are of the form

$$\langle j|\xi\rangle_s = \xi^{-j}\,\mathrm{e}^{-\frac{sj^2}{2}}, \tag{5.3}$$

$${}_s\langle\xi|\eta\rangle_s = \sum_{j=-\infty}^{\infty}(\xi^*\eta)^{-j}\,\mathrm{e}^{-sj^2} = \theta_3\left(\frac{\mathrm{i}}{2\pi}\ln\xi^*\eta\middle|\frac{\mathrm{i}s}{\pi}\right). \tag{5.4}$$

Now, we define the generalized q-deformed coherent states for the quantum mechanics on a circle as the solution of the eigenvalue equation

$$Z_q(s)|\xi\rangle_{s,q} = \xi_q|\xi\rangle_{s,q}, \tag{5.5}$$

where

$$Z_q(s) := \exp(\mathrm{i}(\hat{\varphi} + \mathrm{i}sJ_q)) = \exp\left(\frac{s}{1-q}\left(\frac{1-q^{-1}}{\ln q} - 1\right)\right)\exp\left(\frac{s(q^{-1}-1)}{\ln q}J_q\right)U. \tag{5.6}$$

Of course, the states $|\xi\rangle_q$ given by (3.7) refer to the case with $s = 1$. Using (5.5) and (5.6) we easily obtain the following generalizations of relations (3.10), (3.13) and (3.15), respectively,

$$\langle j|\xi\rangle_{s,q} = \xi_q^{-j}\exp\left(-\frac{s}{\ln q}\frac{q^j-1}{q-1}\right)\exp\left(-\frac{sj}{1-q}\right), \tag{5.7}$$

$$\begin{aligned}{}_{s,q}\langle l,\alpha|h,\beta\rangle_{s,q} = \sum_{j=-\infty}^{\infty}&\exp(\mathrm{i}(\alpha-\beta)j)\exp\left(-\frac{2s}{\ln q}\frac{q^j-1}{q-1}\right)\\ &\times\exp\left(-\frac{q^l+q^h}{1-q}j\right)\exp\left(\frac{2(1-s)}{1-q}j\right),\end{aligned} \tag{5.8}$$

$${}_{s,q}\langle l,\alpha|l,\alpha\rangle_{s,q} = \sum_{j=-\infty}^{\infty}\exp\left(-\frac{2s}{\ln q}\frac{q^j-1}{q-1}\right)\exp\left(-\frac{2q^l}{1-q}j\right)\exp\left(\frac{2(1-s)}{1-q}j\right). \tag{5.9}$$

Applying the d'Alembert ratio test we find that the series (5.9) is not convergent for arbitrary (positive) s. Namely, it follows that it is convergent if $q^l > 1-s$, and divergent if $q^l < 1-s$. Note that these conditions seem to distinguish the case $s = 1$ discussed earlier, because only if $s = 1$ the series (5.9) is convergent for arbitrary l.

6. Discussion

In this work, we have introduced the q-deformed coherent states for the quantum mechanics on a circle. The correctness of the construction is confirmed by the quasi-classical character of the coherent states manifested, for example, by the behaviour of the expectation values of the deformed angular momentum operator. It is worthwhile to recall that the non-deformed coherent states specified by (2.1) as well as the coherent states of a quantum particle on a sphere introduced by us in [9] are concrete realization of the general mathematical scheme of construction of the Bargmann spaces introduced in the recent papers [11–13]. Thus, bearing in mind the observations presented herein, an interesting problem naturally arises of finding deformations of coherent states for the quantum mechanics on a sphere. It should be noted that in view of the observations of Dimakis and Müller-Hoissen [14] one can relate, by a suitable change of variables, the q-deformation of the quantum mechanics on a circle described by the Jackson derivative (3.5) with a discrete quantum mechanics on a lattice. We finally remark that the results of this paper would be of importance in the theory of special functions. We only recall that formulae (3.13), (3.15) and (3.17) describe a quantum deformation of the Jacobi theta functions.

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